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J. Math. Anal. Appl. 299 (2004) 300–304

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

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Weak-mixing implies sensitive dependence ☆

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Received 20 June 2003

Available online 2 September 2004

Submitted by C.E. Wayne

Abstract

Let X be a metric space, $\mathcal{B}$ the σ-algebra of Borel subsets of X, and μ a probability measure on $(X, \mathcal{B})$. In this note, for a measure-preserving map T (respectively a measure-preserving semi-flow φ) on $(X, \mathcal{B}, \mu)$, we prove that if $\operatorname{supp}\mu = X$, and T (respectively φ) is weak-mixing, then T (respectively φ) has sensitive dependence.

Keywords: Measure-preserving map; Measure-preserving semi-flow; Weak-mixing; Sensitive dependence

1. Introduction

Suppose (X, d) is a metric space with a metric d. Write $Z^+ = \{0, 1, 2, \ldots\}$ and $R^+ = [0, +\infty)$. Let $\mathcal{B}(X)$ denote the σ-algebra of Borel subsets of X, and μ be a probability measure on $(X, \mathcal{B})$. Throughout the paper, T is a measure-preserving map on $(X, \mathcal{B}, \mu)$, i.e., for any $B \in \mathcal{B}$ we have $\mu(B) = \mu(T^{-1}B)$, and φ is a measure-preserving semi-flow on $(X, \mathcal{B}, \mu)$, i.e., φ is a semi-flow on X, and for any $B \in \mathcal{B}$ and $t \in R^+$ we have $\mu(B) = \mu(\varphi_t^{-1}B)$, and $\operatorname{supp}\mu = X$.

- T (respectively φ) is called weak-mixing if for any $A, B \in \mathcal{B}$, we have

☆ Project Supported by National Natural Science Foundation of China (10371030).
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.06.066

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\left|\mu(A\cap T^{-i}B)-\mu(A)\mu(B)\right|=0$$
$$\left(\text{respectively}\lim_{t\to\infty}\frac{1}{t}\int_0^t\left|\mu\left(A\cap\varphi_s^{-1}B\right)-\mu(A)\mu(B)\right|ds=0\right).$$

- T (respectively φ) has sensitive dependence if there exists $\delta>0$ such that for any $x\in X$ and any open neighborhood V_x of x, there exists $n\in Z^+$ (respectively $t\in R^+$) such that

$$\sup\left\{d(T^nx,T^ny)\colon\ y\in V_x\right\}>\delta$$
$$\left(\text{respectively }\sup\left\{d(\varphi_tx,\varphi_ty)\colon\ y\in V_x\right\}>\delta\right).$$

- A subset $S\subset Z^+$ (respectively $S\subset R^+$) is called relatively dense, if there exists $N\in Z^+$ (respectively $L\in R^+$) such that for any $k\in Z^+$ (respectively $t\in R^+$) we have

$$S\cap\{k,k+1,\dots,k+N-1\}\neq\emptyset$$
$$\left(\text{respectively }S\cap(t,t+L)\neq\emptyset\right).$$

- A subset $S\subset Z^+$ (respectively Lebesgue measurable set $S\subset R^+$) is called the positive upper density if

$$\limsup_{k\to\infty}\frac{1}{k+1}\operatorname{Card}\{0\leqslant j\leqslant k\colon\ j\in S\}>0$$
$$\left(\text{respectively}\limsup_{t\to\infty}\frac{1}{t}l\left(S\cap[0,t]\right)>0\right),$$

where $l(S)$ is Lebesgue measure of S.

It is well known that sensitive dependence characterizes the unpredictability of chaotic phenomenon. The dependence is the essential condition of various definitions of a system to be chaotic. Therefore, when does a system have sensitive dependence? This question has gained some attention in more recent papers, for example, [1–4]. The authors in [4] proved the result as follows.

Proposition. *Suppose* $\operatorname{supp}\mu=X$, T *is weak-mixing, and satisfies the property*

(P) *for any non-empty open set* $U\subset X$, *there exists a sequence* $\{n_k\}$ *with positive upper density such that* $U\cap(\bigcap_{k\geqslant 0}T^{-n_k}U)\neq\emptyset$,

then T *has sensitive dependence.*

The first aim in this paper is to show that the proposition can be improved by using Khintchine's theorem in [5], i.e., we prove the following

Theorem A. *If* T *is weak-mixing, then* T *has sensitive dependence.*

The second aim is to show that the similar result for measure-preserving semi-flows can be obtained by using corresponding Khintchine's theorem in [6], i.e., we prove the following

Theorem B. *If φ is weak-mixing, then φ has sensitive dependence.*

2. Proof of theorem A

Lemma 2.1 (Khintchine's theorem [5]). *For any $B \in \mathcal{B}$, if $\mu(B) > 0$, then for any $\varepsilon > 0$, the set $\{k \in Z^+\colon \mu(B \cap T^{-k}B) \geqslant \mu(B)^2 - \varepsilon\}$ is relatively dense in Z^+.*

Lemma 2.2. *If T has not sensitive dependence, then there exist two non-empty disjoint open sets U and V in X such that the set $S = \{k \in Z^+\colon T^k V \cap U = \emptyset\}$ has positive upper density.*

Proof. Choose $\delta > 0$ such that for every $x \in X$, there exists $y \in X$ satisfying $d(x, y) > 4\delta$. Since T has not sensitive dependence, there is a point $x \in X$ and a neighborhood V_x of x such that

$$\operatorname{diam}(T^n V_x) = \sup\bigl\{d(T^n x', T^n y')\colon x', y' \in V_x\bigr\} \leqslant 2\delta \quad (\forall n \geqslant 0).$$

Take $0 < \varepsilon < \delta$ such that $B(x, \varepsilon) = \{y \in X\colon d(x, y) < \varepsilon\} \subset V_x$. As $\operatorname{supp}\mu = X$, we have $\mu(B(x, \varepsilon)) > 0$. According to Lemma 2.1, $S = \{k \in Z^+\colon B(x, \varepsilon) \cap T^{-k}B(x, \varepsilon) \neq \emptyset\}$ is relatively dense in Z^+. Now for any $k \in S$, we can find $z \in B(x, \varepsilon) \cap T^{-k}B(x, \varepsilon)$, then $T^k z \in B(x, \varepsilon) \cap T^k B(x, \varepsilon)$, and for any $y \in B(x, \varepsilon)$, we have

$$d(T^k y, x) \leqslant d(T^k y, T^k z) + d(T^k z, x) \leqslant 2\delta + \varepsilon < 4\delta.$$

This shows $T^k B(x, \varepsilon) \subset B(x, 4\delta)$, $\forall k \in S$.

Now let $V = B(x, \varepsilon)$, $U = X \setminus \overline{B(x, 4\delta)}$, then for any $k \in S$ we have $T^k V \cap U = \emptyset$. Moreover, since S is relatively dense, we can find $N \geqslant 1$ such that for any $i \geqslant 0$ we have

$$S \cap \{i, i+1, \ldots, i+N-1\} \neq \emptyset.$$

Consequently

$$\limsup_{m\to\infty} \frac{1}{m+1} \operatorname{Card}\{0 \leqslant k \leqslant m\colon k \in S\} \geqslant \frac{1}{N} > 0. \qquad \Box$$

Proof of Theorem A. If the conclusion does not hold, then by Lemma 2.1, there exist non-empty open sets $U, V \subset X$ such that $S = \{k \in Z^+\colon T^k V \cap U = \emptyset\}$ has positive upper density. Also since $V \cap T^{-k}U \subset T^{-k}(T^k V \cap U) = \emptyset$ $(\forall k \in S)$, so $\mu(V \cap T^{-k}U) = 0$ $(\forall k \in S)$.

Hence,

$$\sum_{k=0}^{n-1} \bigl|\mu(V \cap T^{-k}U) - \mu(V)\mu(U)\bigr| \geqslant \sum_{k=0,\, k\in S}^{n-1} \mu(V)\mu(U)$$
$$= \operatorname{Card}\{0 \leqslant k \leqslant n-1\colon k \in S\}\mu(V)\mu(U).$$

As $\operatorname{supp}\mu = X$, and V, U are all open sets, we know $\mu(V)\mu(U) > 0$. This leads to

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\bigl|\mu(V\cap T^{-k}U - \mu(V)\mu(U)\bigr|$$
$$\geqslant \lim_{n\to\infty}\frac{1}{n}\operatorname{Card}\{0\leqslant k\leqslant n-1\colon\ k\in S\}\mu(V)\mu(U) > 0.$$

This contradicts the weak-mixing of T. $\square$

3. Proof of Theorem B

Lemma 3.1 (Khintchine's theorem [6]). *If for $B \in \mathcal{B}$, we have $\mu(B) > 0$, then the set* $\{t \in R^+\colon\ \mu(B\cap\varphi_t^{-1}B) > \lambda\mu(B)^2\}$ $(\lambda < 1)$ *is relatively dense in R^+.*

Here we need to point out that Khintchine's theorem in [6] is given for measure-preserving flows on $(X, \mathcal{B}, \mu)$, but we can see from its proof that it also holds for measure-preserving semi-flows.

Lemma 3.2. *If φ has not sensitive dependence, then there exist open sets $V, U \subset X$ such that the set*

$$S := \bigl\{t\in R^+\colon\ V\cap\varphi_t^{-1}U = \emptyset\bigr\}$$

has positive upper density.

Proof. Choose $\delta > 0$ such that for any $x \in X$, there exists $y \in X$ satisfying $d(x, y) > 8\delta$. Since φ has not sensitive dependence, there is $0 < \sigma < \delta$ and $x \in X$ such that $\operatorname{diam}(\varphi_t(B(x,\sigma))) < \delta$ $(\forall t \geqslant 0)$. Take $0 < \varepsilon < 1$ such that

$$\varphi_t\bigl(B(x,2\delta)\bigr) \subset B(x,8\delta), \quad \forall t\in[0,\varepsilon].$$

Let $V = B(x,\sigma)$, as $\operatorname{supp}\mu = X$, so $\mu(V) > 0$. According to Lemma 3.1, the set

$$S_1 := \{t\in R^+\colon\ V\cap\varphi_t V\neq\emptyset\} \supset \bigl\{t\in R^+\colon\ V\cap\varphi_t^{-1}V\neq\emptyset\bigr\}$$

is relatively dense in R^+. Hence there exists $L > 0$ such that for any $t \in R^+$ we have

$$S_1\cap(t, t+L)\neq\emptyset.$$

Now for any $t \in S_1$, because $V\cap\varphi_t V \neq \emptyset$, and $\operatorname{diam}(\varphi_t(V)) < \delta$, we have $V\cup\varphi_t(V) \subset B(x, \sigma+\delta) \subset B(x, 2\delta)$, and therefore

$$\varphi_\tau(V)\subset B(x,8\delta), \quad \forall\tau\in[t,t+\varepsilon].$$

Let $U := X\setminus\overline{B(x,8\delta)}$, $S_2 := \bigcup_{t\in S_1}[t, t+\varepsilon]$, then for any $\tau \in S_2$ we have $U\cap\varphi_\tau(V) = \emptyset$, i.e., $V\cap\varphi_\tau^{-1}U = \emptyset$. Consequently $S_2 \subset S$, and

$$\begin{aligned}\limsup_{t\to+\infty}\frac{1}{t}l\big(S\cap[0,t]\big) &\geqslant \limsup_{t\to+\infty}\frac{1}{t}l\big(S_2\cap[0,t]\big)\\ &\geqslant \limsup_{k\to+\infty}\frac{1}{k(L+1)}l\big(S_2\cap\big[0,k(L+1)\big]\big)\\ &\geqslant \frac{k\varepsilon}{k(L+1)}=\frac{\varepsilon}{L+1}>0. \qquad \square\end{aligned}$$

Proof of Theorem B. If the conclusion does not hold, then by Lemma 3.2, there exists non-empty open sets $U, V \subset X$ such that $S := \{t \in R^+\colon\ V \cap \varphi_t^{-1}U = \emptyset\}$ has positive upper density. Also as $\operatorname{supp}\mu = X$, so $\mu(V)\mu(U) > 0$. Therefore

$$\begin{aligned}&\lim_{t\to+\infty}\frac{1}{t}\int_0^t \big|\mu\big(V\cap\varphi_s^{-1}U\big)-\mu(V)\mu(U)\big|\,ds=0\\ &\quad\geqslant \mu(V)\mu(U)\limsup_{t\to+\infty}\frac{1}{t}l\big(S\cap[0,t]\big)>0.\end{aligned}$$

This contradicts the weak-mixing of φ. $\square$

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